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Note on Induced Linear Substitutions.

BY F. FRANKLIN.

The theorem concerning linear substitutions contained in this note is apparently unknown, though its demonstration is extremely obvious.

Let S be a linear substitution operating on the set of variables $(x_1', x_2', \ldots x_\nu')$, converting them into the set $(x_1, x_2, \ldots . x_\nu)$. Then if $a, b, \ldots .$ are the coefficients in the general quantic of the n^{th} degree, $\phi(x_1, x_2, \ldots . x_\nu)$, while $a', b', \ldots .$ are the coefficients in $\phi(x_1', x_2', \ldots .)$, the set $a, b, \ldots .$ is transformed by a certain linear substitution Σ into the set $a', b', \ldots .$ The substitution Σ may be said to be *induced* on the coefficients of an n^{ic} by the substitution S operating on the variables. It is to be noted that S operates upon the *new variables*, while Σ operates upon the *old coefficients.*

Let D be the determinant of the substitution S, Δ that of the substitution Σ; and let D_λ, Δ_λ be the determinants obtained by subtracting λ from each of the diagonal constituents in D, Δ respectively, so that $D_\lambda = 0$, $\Delta_\lambda = 0$, are the characteristic equations of the substitutions S, Σ.

It is evident that *the vanishing of* Δ_λ for a particular value of λ *is the necessary and sufficient condition that there exist an* n^{ic}, ϕ, *which is transformed into* λ *times itself* by the substitution Σ operating on the coefficients, or, what is the same thing, *by the substitution* S *operating on the variables.* But if $\lambda_1, \lambda_2, \ldots . \lambda_\nu$ be the roots of $D_\lambda = 0$, there are ν linear functions $L_1, L_2, \ldots . L_\nu$ which are transformed into $\lambda_1 L_1, \lambda_2 L_2, \ldots . \lambda_\nu L_\nu$ respectively, by the substitution S; hence the various powers and products of the L's of the n^{th} degree are actually n^{ics} which are transformed into themselves multiplied by the corresponding powers and products of the λ's. Hence the powers and products of $\lambda_1, \lambda_2, \ldots . \lambda_\nu$, of the n^{th} degree, are roots of $\Delta_\lambda = 0$; and since the number of these roots, being precisely the number of coefficients in the general n^{ic}, is equal to the degree of Δ_λ, these are *all* the roots of $\Delta_\lambda = 0$. It has here been tacitly supposed that the roots $\lambda_1, \lambda_2, \ldots . \lambda_\nu$ are all distinct; but evidently the result still holds if there are any number of equalities among them. For we have, when the roots are unequal,

$$\Delta_\lambda = (\lambda_1^n - \lambda)(\lambda_2^n - \lambda) \ldots . ,$$

and the right member, being a symmetric function of the roots of D_λ, is a rational entire function of the coefficients of D_λ. The two members of the equation are therefore rational entire functions of the constituents of D which are certainly always equal unless a certain relation is satisfied by these constituents; but this cannot be unless they are identically equal. We have thus proved the theorem:

If the substitution S, viz. $S(x'_1, x'_2, \ldots.) = (x_1, x_2, \ldots.)$, induces on the coefficients of the general n^{ic} the substitution Σ, viz. $\Sigma(a, b, \ldots.) = (a', b', \ldots)$, *the roots of the characteristic equation of the induced substitution are the powers and products, of the* n^{th} *degree, of the roots of the characteristic equation of the original substitution.*

An even simpler, though in a certain sense less elementary, proof of the theorem may be obtained by supposing S transformed by a linear substitution U into the canonical form, in which only the diagonal constituents appear; the induced substitution Σ will evidently be simultaneously transformed by some substitution Υ. Thus we have

$$U^{-1}SU(x'_1, x'_2, \ldots. x'_\nu) = (\lambda_1 x'_1, \lambda_2 x'_2, \ldots. \lambda_\nu x'_\nu) = (x_1, x_2, \ldots. x_\nu),$$

whence $$ax_1^n + bx_2^n + \ldots. = a\lambda_1^n x_1^n + b\lambda_2^n x_2^n + \ldots.,$$

so that $$\Upsilon^{-1}\Sigma\Upsilon(a, b, \ldots.) = (\lambda_1^n a, \lambda_2^n b, \ldots.).$$

Now the characteristic equation of a linear substitution is unaltered by transformation; hence the roots of the characteristic equation of S are $\lambda_1, \lambda_2, \ldots. \lambda_\nu$ and those of Σ are $\lambda_1^n, \lambda_2^n, \ldots.$ Q. E. D.

The theorem may evidently be extended to the substitution Σ induced on the coefficients of a quantic in several sets of variables $x_1, x_2, \ldots. x_\nu$; $y_1, y_2, \ldots. y_{\nu'}$; $z_1, z_2, \ldots. z_{\nu''}$; $\ldots.$ by substitutions $S, S', S'', \ldots.$ affecting these sets respectively. If the quantic is of degrees $n, n', n'', \ldots.$ in the x's, the y's, the z's, $\ldots.$ respectively, *the roots of the characteristic equation of* Σ *are all those products of powers of the roots of the characteristic equations of* $S, S', S'', \ldots.$, *which are of the degree* n *in the first set of roots,* n' *in the second set, etc.* To take a single example: Let u_1, u_2, u_3 be contragredient to x_1, x_2, x_3 if a substitution giving the roots $\lambda_1, \lambda_2, \lambda_3$ be performed upon the x's, the roots corresponding to the substitution on the u's are $1/\lambda_1, 1/\lambda_2, 1/\lambda_3$; then if Σ denote the substitution induced upon the coefficients of

$$a_{11}x_1u_1 + a_{12}x_1u_2 + \ldots. + a_{33}x_3u_3,$$

the roots of the characteristic equation of Σ are

$$1, 1, 1, \lambda_1/\lambda_2, \lambda_2/\lambda_1, \lambda_1/\lambda_3, \lambda_3/\lambda_1, \lambda_2/\lambda_3, \lambda_3/\lambda_2.$$

Finally, it should be observed that the theorem here presented evidently furnishes us with a *sufficient* as well as necessary condition that a proposed substitution be transformable into an induced substitution. *E. g.*, a substitution in 6 variables is transformable into an induced substitution if the roots of its characteristic equation are the squares and products of three quantities.